1 Cooperative bargaining

1.1 The bargaining problem

Bargaining situations are ubiquitous in our times, both in the Western and the Eastern hemisphere. Wage negotiations between a group of employers and a trade union, trade agreements between single countries (e.g. the US and Mexico) or between larger associations (the European Union and the US) or, in the political sphere, disarmament talks between East and West during the cold war era, and, last but not least, environmental negotiations among developed nations and between developed and less developed countries are only some examples of bargaining that have received considerable attention over the years. The problem is one of a choice of a feasible alternative by a group of people or nations or associations with often conflicting preferences in a framework of cooperation. As Kalai (1985, p. 77) writes, this 'may be viewed as a theory of consensus, because when it is applied it is often assumed that a final choice can be made if and only if every member of the group supports this choice'. Kalai continues saying that 'because this theory deals with the aggregation of peoples' preferences over a set of feasible alternatives, it bears close similarities to theories of social choice and the design of social welfare functions'. The final outcome that the individuals (or groups) involved strive for may be attained by the parties themselves. Sometimes, however, the final result will be reached via the mediation of an outside person, an arbitrator.

There is one feature that distinguishes the bargaining problem fundamentally from almost all the other social choice approaches. It is the existence of a threat or disagreement outcome which comes about when the people involved in bargaining fail to reach an agreement. Consequently, the gains that the bargainers (may) achieve are evaluated or measured with reference to the disagreement point.

A particularly vivid and perspicuous situation was discussed by Braithwaite (1955). Luke and Matthew occupy two flats in a house. There is no third party around. Unfortunately, the acoustics in this house are such that each of the two men can hear everything louder than a conversation that takes place in the other person's flat. Luke likes to play classical music on the piano, Matthew likes to improvise jazz on the trumpet. It just happens to be the case that each of the two men has only the hour from 9 to 10 in the evening for recreation, i.e., to play music, and that it is impossible for either to change to another time. Suppose that the satisfaction of each man from playing his instrument for the hour is affected by whether or not the other is also playing. More explicitly, Luke, the pianist, prefers most that he play alone, next that Matthew play alone, third that neither play, and finally that both play. Matthew, the trumpeter, prefers most that he play alone, then that Luke play alone, third that they both play at the same time, and last that neither play.

The solution that Braithwaite suggests for this conflict situation allocates substantially more time (in terms of number of evenings per month, let's say, though the author's analysis is in utility values) to the jazz trumpeter than to the classical pianist. The reason for this lies in the diverging preferences of the two men. Matthew has a threat advantage before there is a contract since he prefers that both of them play at the same time to neither of them playing while Luke's preferences on these two outcomes are just the opposite.

There is another feature that distinguishes bargaining theory from much of social choice theory. It is the fact that in bargaining problems physical outcomes or objects that are to be distributed (such as commodity bundles and/or amounts of labour to be provided) are almost entirely ignored. What matters are the utility combinations of the agents, more precisely their net–gains over the starting point or status quo. Any two bargaining situations are considered the same whenever they are described by the same set of feasible utility vectors. This comes very close to the diagrammatic analysis in chapter 7.4 where, as the reader will remember, only utility allocations mattered. We called this approach welfaristic. Bargaining theory also evolves in a welfaristic framework.

1.2 Nash's bargaining solution

We shall start by introducing a certain amount of formalism that will be valid throughout this chapter. Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents or players and let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set of physical objects or social outcomes. In chapter 7.5, we called the latter pure prospects and we introduced the notion of a lottery $\mathbf{p} = (p_1, \ldots, p_m)$ that offers the pure prospect x_i with probability p_i. We shall do exactly the same here and define the set of all lotteries L as $L = \{\mathbf{p} \in \mathbb{R}^m | p_i \geq 0$ for all i and $\sum_{i=1}^{m} p_i = 1\}$. So if there are two prizes in case of a lottery, a weekend in a nice hotel (occurring with probability p) and 300 Euros in cash (occurring with probability $1 - p$), an increase or decrease of p is tantamount to saying that the weekend becomes more or less likely as the final outcome of this lottery. A second example of such convex combinations is further away from the notion of a lottery and closer to the concept of a 'mixed alternative'. If in case of a legacy, there is a house in the city (x_i) and a flat at the sea (x_j), two children who inherited these two indivisible objects from their deceased parents could decide that each gets $px_i + (1 - p)x_j$ with $p = 1/2$, which means that each child obtains the right to spend half a year in the house in the city and half a year in the flat at the sea. By varying p, all points along the straight line between x_i and x_j become feasible. The formal consequence of permitting probability combinations among any number of discrete objects is that a convex space is obtained. In other words, the space of all lotteries or mixed alternatives is convex. This fact is of importance for the analysis that follows. Furthermore, we wish to assume that the space of lotteries is compact which means that this space is closed and bounded.

Also in the current chapter (as in chapter 7.5), it is postulated that all agents possess a von Neumann–Morgenstern utility function. Each player will assess the utility of any lottery $\mathbf{p} \in L$ in terms of expected utility. If u_k is agent k's cardinal utility function on the space of lotteries and agent k obtains the mixed alternative or option $[px_i, (1-p)x_j]$, then the utility of this mixed alternative according to expected utility theory is $u_k([px_i, (1-p)x_j]) = p \cdot u_k(x_i) + (1-p) \cdot u_k(x_j)$.

Given the set X of social states and the convex space of lotteries L, we still have to introduce $x_0 \in L$, the status quo or threat point in case of disagreement. It could, for example, be a historically given point before bargaining starts and/or a point to which the agents revert in case there will be no bargaining agreement. Next, we use the set of individual von Neumann–Morgenstern utility functions, one utility function for each of the n individuals where each function is determined only up to some positive affine transformation (this is the case of cardinal measurability without interpersonal comparability, abbreviated by CMN in chapter 7.2 on the informational structure of utility representations). Applying these utility functions, we map the space L together with the threat point x_0 into a feasible set S of utility vectors and the corresponding disagreement point d. Since L was assumed to be convex and compact, the feasible set S will inherit these properties, given the n von Neumann–Morgenstern utility functions.

Let us now consider the following two definitions.

Definition 8.1. A pair (S, d) with $S \subseteq \mathbb{R}^n$ and $d \in S$ is called a bargaining situation for n persons, if S is convex and compact and there is at least one $s \in S$ with $s > d$.

The last requirement $s > d$ means that for each of the n individuals, there is a genuine incentive to reach an agreement. It is individually rational for each person to reach and support such an agreement.

Let B^n be the set of all bargaining situations with n agents.

Definition 8.2. A function $f\colon B^n \to \mathbb{R}^n$ such that for every $(S, d) \in B^n$, $f(S, d) \in S$ is a bargaining solution. The coordinates of the solution are $f_1(S, d), \ldots, f_n(S, d)$.

Figure 8.1 illustrates the two definitions in 2–dimensional utility space.

Figure 8.1 about here

Nash (1950) had a unique solution to the bargaining problem in mind. It should be a 'fair bargain' in a situation where the individuals are highly rational, are equal in bargaining skill and 'each has full knowledge of the tastes and preferences of the other' (1950, p. 155). Nash formulated four requirements to be satisfied by $f(S,d)$.

Axiom 1 (Invariance with Respect to Utility Transformations). Given a bargaining situation $(S,d) \in B^n$ and 2 n numbers $a_1 > 0, \ldots, a_n > 0$ and $b_1, \ldots, b_n$, let the bargaining situation (S', d') be defined by $S' = \{y \in \mathbb{R}^n | \exists\, x \in S : \; y_i = a_i x_i + b_i\}$, $d'_i = a_i d_i + b_i$ for all $i \in \{1, \ldots, n\}$. Then $f_i(S', d') = a_i f_i(S,d) + b_i$.

This means that if the agents transform their utility scales independently, the n coordinates of the solution will change by the same utility transformations. Let us consider the following example in $\mathbb{R}^2_+$ (Figure 8.2a). There is the status quo point $d = (0,0)$, point $\bar{u} = (0, \bar{u}_2)$, point $\bar{\bar{u}} = (\bar{\bar{u}}_1, 0)$, moreover all utility allocations as convex combinations between d and $\bar{u}$, d and $\bar{\bar{u}}$ and between $\bar{u}$ and $\bar{\bar{u}}$, and all points in the interior of this triangle. All these points together constitute bargaining situation (S,d) in $\mathbb{R}^2_+$. Let solution $f(S,d)$ be the point halfway on the hypotenuse of this triangle. We now choose the following transformations of the two utility functions: $u'_1(\cdot) = a_1 u_1(\cdot) + b_1$, $u'_2(\cdot) = a_2 u_2(\cdot) + b_2$. Let $a_1 = 2$ and $a_2 = \frac{1}{2}$. Then $d = (0,0)$ is transformed into $d' = (b_1, b_2)$, $\bar{u}$ is transformed into $\bar{u}' = (b_1, \frac{1}{2}\bar{u}_2 + b_2)$, and $\bar{\bar{u}}$ is transformed into $\bar{\bar{u}}' = (2\bar{\bar{u}}_1 + b_1, b_2)$. If the solution point $f(S', d')$ of the transformed bargaining situation lies halfway on the line between $\bar{u}'$ and $\bar{\bar{u}}'$, i.e., $f(S', d') = (\bar{\bar{u}}_1 + b_1, \frac{1}{4}\bar{u}_2 + b_2)$, then this is exactly what axiom 1 requires. The independent utility transformations must not change 'the character' of the solution point $f(S,d)$.

Figure 8.2.a about here

Figure 8.2.b about here

Axiom 2 (Weak Pareto Efficiency). If $x \in S$ and there exists another point $y \in S$ such that $y > x$ (i.e., $y_i > x_i$ for all i), then $x \neq f(S,d)$.

The solution point $f(S,d)$ always lies on the boundary of the convex set S in the north–east direction.

Axiom 3 (Symmetry). Let (S,d) be a symmetric bargaining situation in B^n, i.e., $d_1 = d_2 = \ldots = d_n$, and for each $x \in S$, every permutation of x is also in S. Then, $f_1(S,d) = f_2(S,d) = \ldots = f_n(S,d)$.

This condition requires that if the persons involved in a bargaining situation are indistinguishable both with respect to their status quo point and their possible utility allocations, the solution will have to be equal for all players. The bargaining situation (S,d) in Figure 8.3.a is symmetric, the one in Figure 8.3.b is not.

Figure 8.3.a about here

In Figure 8.3.a, the solution $f(S, d)$, according to axioms 2 and 3, will lie at the point where the 45^o line through the origin and the hypotenuse of the triangle intersect.

Axiom 3 documents the welfaristic character of bargaining theory very clearly. The only information that counts is the information contained in the pair (S, d). The labeling of the agents and the underlying allocations in the outcome space do not count at all. It can, for example, be the case that in the outcome space, the allocations are not 'symmetric' but that they become symmetric in utility space because of the individually independent utility assessments.

Note that in all cases in which set S is triangular, the solution can be determined via a combination of axioms 3 and 1, together with weak Pareto efficiency.

Axiom 4 (Independence of Irrelevant Alternatives or Contraction Consistency). If (S, d) and (T, d) are bargaining situations in B^n with $S \subset T$ and if, in addition, $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

Figure 8.3.b about here

This fourth condition is very important for the Nash bargaining solution and therefore requires a couple of extra comments. Axiom 4 looks at any two bargaining situations with an identical status quo point d. Both situations differ in so far as certain points which are feasible in T are no longer possible when the set shrinks from T to S. The axiom requires that the solution for T should also be the solution for S, if $f(T, d)$ is a point in S. The fact that certain utility vectors are no longer possible should not matter in such cases. Nash's fourth requirement can be interpreted as a rationality condition under set contraction. In the social choice literature, this axiom is known as property α (though Nash's requirement is confined to the choice of unique points). It was defined and discussed in chapter 1. Figure 8.4 depicts the consistency issue.

Axiom 4 can also be viewed in a different, though analogous way. Starting with a feasible set S and its solution $f(S, d)$, the choice from the larger set T when some new alternatives are added should either be $f(S, d)$, the 'old' choice, or one of the new alternatives. The choice of the name 'independence of irrelevant alternatives' which was not given by Nash himself but by Luce and Raiffa (1957) and others, was very unfortunate indeed because of Arrow's condition of the same name. At the beginning, quite a few scholars thought that Arrow's and Nash's condition meant the same. As the reader who studied chapter 2 above, of course, knows, this conjecture is totally false.

We can now formulate Nash's theorem for bargaining situations.

Figure 8.4 about here

THEOREM 8.1 There is exactly one bargaining solution on B^n that satisfies axioms 1–4. It is the function F with $F(S,d) = x$ such that $x > d$ and $\prod_{i=1}^{n}(x_i - d_i) > \prod_{i=1}^{n}(y_i - d_i)$ for all $y \in S$ with $y > d$ and $y \neq x$.

F is called the cooperative Nash solution. F maximizes, in the individually rational region of S ($s_i \geq d_i$ for all i), the product of utility gains over the status quo point d.

In what follows, we shall present a proof of this result in $\mathbb{R}^2$. Proofs have been given at various places in the literature. First of all, there is Nash's (1950) original proof. Similar proofs can be found in Luce and Raiffa (1957, chapt. 6.5), Harsanyi (1977, chapt. 8.3), Roemer (1996, chapt. 2.2), and others.

Proof. It is easily seen that F satisfies the four axioms. Let us consider the other direction. We begin with any bargaining situation (S,d) and denote by x the point selected by the Nash formula on S. Such a point will always exist because of compactness of S, and since S is convex, this point will also be unique. We now apply axiom 1 and transform (S,d) into (S',d') such that $ad + b = d' = (0,0)$ and $ax + b = (1,1)$, where $a_1 > 0,\ a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$.

We next show that (1,1) is the solution of $(S',(0,0))$. The Nash formula took its maximum value on S at point x; this property is preserved under positive affine transformations. In other words, (1,1) maximizes the product $s_1' \cdot s_2'$ in S'. The point (1,1) clearly lies on the line $s_1' + s_2' = 2$. This line which has the character of a hyperplane separates the rectangular hyperbola $s_1' \cdot s_2'$ from the convex set S' at the point (1,1). This is shown in Figure 8.5. We now construct a triangle ABC containing $d' = (0,0)$ which is symmetric with respect to the 45^o line between d' and (1,1). We call this bargaining situation $(T,(0,0))$. It fully contains set S'. By axioms 2 and 3, since T is symmetric, the solution must be (1,1). Consequently, by axiom 4, the point (1,1) must also be the solution for (S',d'), i.e., $f(S',d') = (1,1)$. But then, by invariance axiom 1, $f(S,d) = x$ for the original bargaining situation, which completes the proof.

Figure 8.5 about here

For a two–person society (as in our proof above), the Nash solution can be illustrated in the following geometrical way. Compare the areas of all rectangles which have as their common 'south–west' corner point d and each of them has as its 'north–east' corner a point on the boundary of S. Look for that point on the boundary of S where the area of the corresponding rectangle is maximal or where the product of coordinates among the Pareto efficient points is a maximum. The coordinates of this point represent the Nash bargaining solution. Rectangles in geometry represent extensions in two dimensions. Here, the dimensions are the utilities of two persons.

Various objections have been raised against Nash's bargaining solution and the underlying axioms. One criticism deals with the fact that Nash's approach does not allow for interpersonal comparisons of utility. Since in the CMN set–up, utility scales and origins can be chosen and varied independently among agents, there is no hope for any 'degree' of interpersonal comparability. Does this then imply that such a framework is totally inappropriate for questions of distributive justice?

Another objection refers to the fact that Nash's solution crucially depends on the position of the status quo point. A favourable coordinate as well as a not–so–favourable coordinate in the vector of status quo utilities are well reflected in the final bargaining solution à la Nash. Since the status quo indicates the threat potentials or the strength of the different agents involved in the bargaining procedure, the question arises whether the Nash approach can generate a solution that is ethically appealing. Rawls's (1971) answer to this question is clear. Rawls's two principles of justice were the object of a collective agreement under a veil of ignorance. Gauthier, another philosopher, saw this point differently. Natural differences such as talents should not be viewed as arbitrary. They should count and enter into the description of a status quo. Society should not redress them (Gauthier, 1978). Rawls deliberately prevented certain types of information to enter an agreement over basic principles for society. 'To each according to his or her threat advantage' would be an unacceptable principle of justice for him.

The argument that the Nash approach only considers utility allocations and ignores the underlying economic environment, i.e., the physical objects to be distributed, has been mentioned before. This objection will apply to the other models of bargaining presented in this chapter as well.

A lot of criticism has been leveled against Nash's independence condition. This criticism actually led to an alternative proposal, to be discussed in the fourth section of this chapter, where Nash's independence axiom is replaced by an axiom of monotonicity. In order to understand and appreciate the criticism against independence, we can go back to Figure 8.4 above. Imagine that 'at the beginning', there is the triangular set T of utility allocations for players 1 and 2. According to the Nash formula, $f(T, d)$ is the solution. Now imagine that the utility possibilities of person 2 shrink so that the new set is the trapezoid S. According to Nash, the solution is exactly the same as before. Is this reasonable? Agent 2 lost some of his or her potential, but apparently, the solution does not reflect this. Agent 2 even gets the maximum possible utility under the new situation. Luce and Raiffa (1957, p. 133) raise the question whether this can be considered as 'fair'. The two authors reverse the procedure, i.e., they start with S and then go to T. Does agent 2 now deserve more? Luce and Raiffa argue that 'the status quo serves to point out that certain aspirations are merely empty dreams' (p. 133), admitting that they have changed their mind on this question in the course of time. We shall revert to this issue in section 8.4 below.

One last remark about the Nash approach. We have just seen that this bargaining solution presupposes cardinal measurability without interpersonal comparability. Sen (1970b, chapter 8) has shown that cardinality without interpersonal comparability leads to an impossibility result that is analogous to Arrow's negative statement. How can this be, i.e., a definitely positive result à la Nash and another negative verdict à la Arrow, all coming about in the same informational set–up? Do we encounter an inconsistency here? No, fortunately not, and the answer is rather simple. An integral part of the Nash solution is the status quo point. In other words, the Nash approach does not satisfy Arrow's independence condition reformulated in terms of cardinal utilities. To be more explicit, the evaluation of two social outcomes x and y, let's say, depends on the position of some third alternative, the status quo point. Thus, Nash's solution can be seen as a possibility result in cardinal utility aggregation.

1.3 Zeuthen's principle of alternating concessions

Zeuthen (1930) proposed a bargaining procedure in which agents make alternating offers. Harsanyi (1977, chapt. 8) has shown that the equilibrium of Zeuthen's approach, though quite different in character from that of Nash himself, constitutes the Nash bargaining solution. In this way, elements of non–cooperative games are introduced that may render the Nash bargaining solution as the predicted outcome of self–interested players more convincing. Here are the details of Harsanyi's analysis.

Zeuthen described collective bargaining on the labour market. Offers were made in terms of monetary units (wages) but without loss of generality, these can be expressed as units of utility. Remember that in the case of Nash's bargaining solution, we started from a set X of social states and the corresponding space of lotteries L and then mapped the latter into a set S of utility vectors. Let there be two agents or players 1 and 2. At a certain point in time, both players propose a Pareto–efficient agreement. Let us assume that agent 1 proposes $x = (x_1, x_2)$ and agent 2 proposes $y = (y_1, y_2)$, where the first component always refers to player 1. If the agents fail to reach an agreement, a conflict situation arises which gives, in terms of utilities, $u_1(x_{01})$ to agent 1 and $u_2(x_{02})$ to agent 2, where $x_0 = (x_{01}, x_{02})$ is the conflict or status quo situation. We assume that $u_1(x_{01}) < u_1(y_1) < u_1(x_1)$ and $u_2(x_{02}) < u_2(x_2) < u_2(y_2)$. In other words, player 1(2) prefers the own proposal to the offer made by player 2(1) but prefers either proposal to the conflict or status quo situation x_0.

What will now happen? If one of the two players accepts the other's proposal, an agreement is reached. If this is not the case, neither for agent 1 nor for agent 2, the conflict situation may arise. Or, at least one of the players comes up with a new proposal (for example, player 1 with a new proposal $x' = (x'_1, x'_2)$) that is more favourable to the other player than the last proposal but less favourable than the other player's own last offer. In other words, any new proposal x' by player 1 will satisfy $u_2(x_2) < u_2(x'_2) < u_2(y_2)$.

At this point, Zeuthen's idea of a concession comes in. If player 1 accepts the offer y made by player 2, the concession of player 1 is $u_1(x_1) - u_1(y_1)$. If player 2 agrees to the proposal x of player 1, the concession of player 2 is $u_2(y_2) - u_2(x_2)$. Zeuthen then asks, given that at a certain stage of mutual offers, an agreement has not yet been reached: 'Which player will (have to) make the next concession?' His answer is that the next concession must always come from the player less willing to face the risk of a conflict, in other words face the point (x_{01}, x_{02}).

How can one measure a given agent's willingness to risk a conflict rather than accept the opponent's terms? Zeuthen proposes that each of the two players basically has two options, viz. to stick to his or her last offer or to accept the opponent's terms. Let player 1's last offer be x and the opponent's last offer be y. If both agents are Bayesian expected–utility maximizers, they assign subjective probabilities to the two possible choices that the other agent can make. Let p_{12} be the subjective probability that 1 assigns to the hypothesis that 2 will stick to his or her last offer, and $(1 - p_{12})$ be the subjective probability that 1 assigns to the hypothesis that player 2 will accept player 1's last offer.

If player 1 accepts the opponent's last offer, player 1 will obtain $u_1(y_1)$. If 1 simply sticks to his or her last proposal, player 1 may obtain $u_1(x_1) > u_1(y_1)$ with probability $(1-p_{12})$, but player 1 may also obtain the lower utility $u_1(x_{01}) < u_1(y_1)$ with probability p_{12}. Consequently, if player 1 maximizes the own expected utility, he or she will stick to the own last offer x only if $(1 - p_{12}) \cdot u_1(x_1) + p_{12} \cdot u_1(x_{01}) \geq u_1(y_1)$. This expression is equivalent to $p_{12} \leq \frac{u_1(x_1) - u_1(y_1)}{u_1(x_1) - u_1(x_{01})}$.

The latter ratio is called player 1's risk limit r_1, since it stands for the highest risk (the highest subjective probability of ending in a conflict situation) that player 1 would be willing to face in order to achieve a settlement according to the own proposal x rather than on the opponent's terms y. With probability p_{12}, player 1 must expect a conflict to occur if he or she sticks to the last own offer. According to the formula above, the highest value of probability p_{12} that 1 can accept without consenting to the opponent's last offer is $p_{12} = r_1$ (note that $0 \leq r_i \leq 1$ for $i \in \{1, 2\}$).

The quantity r_i is the ratio of two utility differences. The numerator, for example $u_1(x_1) - u_1(y_1)$ for agent 1, has already been interpreted as the concession of this player. In Harsanyi's words, it is 'the *cost* to player i *of reaching an agreement on the opponent's terms* instead of an agreement on player i's own terms' (p. 151, the italics are Harsanyi's). The denominator is the cost to agent i if there is no agreement with the opponent. The ratio r_i is 'a measure of the strength of player i's incentives for insisting on his own last offer rather than accepting his opponent's last offer' (p. 151).

As already stated above, the quantity r_i measures the highest risk that agent i is prepared to take rather than to accept his or her opponent's terms. If $r_i < r_j$, player i is less willing than player j is to risk a conflict and therefore has weaker incentives to do so. This information is known to both players. Therefore, there will be a strong incentive for player i to make the next concession. Thus, Zeuthen proposes the following decision rule which Harsanyi calls 'Zeuthen's Principle':

(a) If $r_1 > r_2$, then player 2 has to make the next concession;
(b) If $r_1 < r_2$, then player 1 has to make the next concession;
(c) If $r_1 = r_2$, then both players have to make some concessions.

The player who has to make a concession along the rule above is free to make a quite small concession. However, it should not be smaller than some minimum size ensuring that the agents' alternating offers will converge to some agreement after a finite number of bargaining rounds. This alternating offer bargaining process will eventually reach an agreement that corresponds to the Nash bargaining solution. This will now be shown.

Let us assume that person 1 proposes $x = (x_1, x_2)$, while person 2 proposes $y = (y_1, y_2)$ with

(a) $r_1 = \frac{u_1(x_1)-u_1(y_1)}{u_1(x_1)-u_1(x_{01})} \leq \frac{u_2(y_2)-u_2(x_2)}{u_2(y_2)-u_2(x_{02})} = r_2$.

According to the Zeuthen Principle, agent 1 has to make the next concession. Let us specify this as $x' = (x'_1, x'_2)$ such that

(b) $r'_1 = \frac{u_1(x'_1)-u_1(y_1)}{u_1(x'_1)-u_1(x_{01})} \geq \frac{u_2(y_2)-u_2(x'_2)}{u_2(y_2)-u_2(x_{02})} = r'_2$.

Expression (a) can be shown to be equivalent to

(a') $(u_1(x_1) - u_1(x_{01})) \cdot (u_2(x_2) - u_2(x_{02})) \leq (u_1(y_1) - u_1(x_{01})) \cdot (u_2(y_2) - u_2(x_{02}))$.

Expression (b) can be shown to be equivalent to

(b') $(u_1(y_1) - u_1(x_{01})) \cdot (u_2(y_2) - u_2(x_{02})) \leq (u_1(x'_1) - u_1(x_{01})) \cdot (u_2(x'_2) - u_2(x_{02}))$.

Expression (a') says that the Nash product according to player 1's first offer is smaller or equal to the Nash product according to player 2's first offer. Expression (b') says that the latter is smaller or equal to the revised proposal of the first player. Because of relation (b) and Zeuthen's Principle, the next proposal should be made by player 2, and it is easy to see that it will yield an even higher Nash product. In other words, in each round the offer corresponding to the smaller value of the Nash product will be eliminated, while the offer corresponding to the larger Nash product will be retained until the next round. This alternating procedure will continue until one of the two players makes a proposal that corresponds to the largest possible value of the Nash product. Since no further improvement will be possible from there, both players will accept this offer. In other words, the final point in Zeuthen's alternating sequence will be the point where the Nash product is maximal, and this is, as we know from the last section, the Nash solution point. There are various other models with alternating offers within a non–cooperative set–up that eventually lead to the Nash bargaining solution (see e.g. Rubinstein et al. (1992)).

1.4 The Kalai–Smorodinsky bargaining solution

At the end of chapter 8.2, we discussed an objection against Nash's independence condition that is related to the situation in Figure 8.4, but we also mentioned Luce and Raiffa's change of mind on this issue. Should a player receive more when the set of feasible utility vectors expands in a direction that is favourable to this person?

Let us look at the following situation which is taken from Kalai and Smorodinsky (1975) and is depicted in Figure 8.6. We assume that there are two bargaining situations $(S^1, 0)$ and $(S^2, 0)$ in $\mathbb{R}^2$ with the following characteristics:

$$S^1 = \text{convex hull } \{(0,1),(1,0),(3/4,3/4)\},$$
$$S^2 = \text{convex hull } \{(0,1),(1,0),(0.99,0.7)\},$$

For any given value of u_1 such that $0 < u_1 < \max u_1$, there is a value of u_2 such that $(u_1, u_2) \in (S^2, 0)$ and this utility level for player 2 is strictly higher than the corresponding maximally possible utility value attainable in $(S^1, 0)$. Kalai and Smorodinsky (1975, p. 515) argue that 'based on these facts', player 2 has a good reason to demand that he get more under $(S^2, 0)$ than he does under $(S^1, 0)$. However, the Nash solution which in each of the two bargaining situations lies on the kink does not satisfy player 2's demand.

Figure 8.6 about here

Kalai and Smorodinsky propose to replace Nash's independence condition by an axiom of monotonicity. In order to describe their solution concept, we have to introduce the concept of an 'ideal point'.

Definition 8.3. Given any bargaining situation $(S, d) \in B^2$, the vector $\bar{x}(S, d) = (\bar{x}_1, \bar{x}_2)$ with $\bar{x}_i = \max_{(S,d)} \{s_i|(s_1, s_2) \geq d\}$ for $i \in \{1, 2\}$ is called the ideal point of (S, d).

The ideal point of (S, d) has as its components the maximally possible utility value of each player. In Figure 8.6, the maximally possible utility value of player 1 is $\bar{x}_1 = 1$, for player 2 it is $\bar{x}_2 = 1$ so that $\bar{x}(S, d) = (1, 1)$. The reader should note that very often, the ideal point lies outside the feasible set.

Axiom 5 (Monotonicity). If (S^1, d) and (S^2, d) are two bargaining situations in B^2 such that $S^1 \subseteq S^2$ and $\bar{x}_1(S^1, d) = \bar{x}_1(S^2, d)$, then $f_2(S^1, d) \leq f_2(S^2, d)$. Similarly, if $\bar{x}_2(S^1, d) = \bar{x}_2(S^2, d)$, then $f_1(S^1, d) \leq f_1(S^2, d)$.

This axiom uses the intuition from Figure 8.6 and states that if there is a set expansion from S^1 to S^2 with a fixed status quo d, while the maximally possible utility level for player 1 remains unchanged, player 2's utility level according to $f(S, d)$ should not go down but weakly increase (and analogously, when players 1 and 2 are interchanged).

We have reached some kind of a bifurcation. Axioms 1–3 from section 2 of this chapter together with axiom 4 yield the Nash solution, axioms 1–3 together with axiom 5 above yield the Kalai–Smorodinsky solution .

THEOREM 8.2 There is one and only one solution, μ, on B^2 satisfying axioms 1–3 and the axiom of monotonicity. The solution μ has the following representation. For $(S, d) \in B^2$, construct the line from d to $\bar{x}(S, d)$. The maximal element of S on this line is $\mu(S, d)$.

Note first that this theorem is formulated for bargaining situation in B^2. We shall say more about this point after the proof of this result. Secondly, the solution point on the ray from d to $\bar{x}(S, d)$ can be nicely interpreted in a geometrical way. The Kalai–Smorodinsky solution is the maximal point $\mu(S, d) = x$ in the set of individually rational points such that
$\frac{x_1 - d_1}{\bar{x}_1 - d_1} = \frac{x_2 - d_2}{\bar{x}_2 - d_2}$. If $\frac{x_i - d_i}{\bar{x}_i - d_i}$ is interpreted as a relative utility gain of person i, the Kalai–Smorodinsky solution leads to an equalisation of relative utility gains of the two players. This equalisation holds for every point along the ray between d and $\bar{x}(S, d)$. Again, we obtain a unique bargaining solution (see Figure 8.7).

Figure 8.7 about here

Proof. The proof follows Thomson (1994c) and Roemer (1996). It is clear that the Kalai–Smorodinsky solution μ satisfies the four conditions in the theorem. Conversely, let there be an arbitrary bargaining situation (S, d) in the plane. Due to axiom 1, it is possible to transform (S, d) into (S', d') such that $d' = (0, 0)$ and $\bar{x}(S, d)$ is mapped into (1,1). We call the new bargaining situation $(S', 0)$. Under invariance axiom 1, the solution under f on (S, d) maps into the solution on $(S', 0)$. The solution $\mu(S', 0)$ on

$(S', 0)$ is a point that has equal coordinates because the ray connecting the threat point $(0, 0)$ to the point $\bar{x}(S', 0) = (1, 1)$ has slope one. Let us call this solution point (a, a).

Now construct S'' inside S' in the following way. Connect point (a, a) to the points $(1, 0)$ and $(0, 1)$. S'' will be a four–sided convex set where two sides are segments along the axes and the other two sides are the lines between $(1, 0)$ and (a, a), and between (a, a) and $(0, 1)$ – see Figure 8.8. The bargaining situation $(S'', 0)$ is symmetric. Therefore, due to the Pareto condition and the symmetry axiom, $f(S'', 0) = (a, a)$. Note that $(S'', 0)$ and $(S', 0)$ are related to each other in the way spelled out in the antecedent of axiom 5, i.e., $\bar{x}_1(S'', 0) = \bar{x}_1(S', 0)$ and $\bar{x}_2(S'', 0) = \bar{x}_2(S', 0)$. Therefore, from this axiom, $f_i(S', 0) \geq f_i(S'', 0)$ for $i \in \{1, 2\}$. Thus, $f(S', 0) \geq (a, a)$ and since (a, a) is Pareto–optimal on S', $f(S', 0) = (a, a)$. Notice that (a, a) is the Kalai–Smorodinsky solution on $(S', 0)$. So $f(S', 0) = \mu(S', 0)$. But then, by invariance axiom 1, it follows that $f(S, d) = \mu(S, d)$ for the original bargaining situation, which completes the proof.

Figure 8.8 about here

The Kalai–Smorodinsky solution is well–defined for situations with any finite number of participants. However, this solution does not necessarily satisfy the Pareto efficiency condition for bargaining situations with more than two players. Actually, Roth (1979) has shown that for bargaining situations with three or more participants, no solution exists that fulfils the conditions of Pareto efficiency and symmetry together with monotonicity.

In order to illustrate Roth's negative result, we give the following example which is also due to Roth (1979). Consider a 3–person bargaining situation whose disagreement point is equal to the origin. Let the feasible set S be equal to the convex hull of $d = (0, 0, 0)$ and the two points (1,0,1) and (0,1,1). Clearly, the set of Pareto efficient points in S is the line segment joining (1,0,1) and (0,1,1). Any solution $f(S, d)$ that is to satisfy Pareto efficiency has to allocate one unit of utility to person 3. The ideal point of this game is $\bar{x}(S, d) = (1, 1, 1)$. For this set (see Figure 8.9), the Kalai–Smorodinsky solution is $\mu\,(S, d) = (0, 0, 0)$ which collides heavily with Pareto efficiency. This solution is in fact dominated by all other points of S.

Figure 8.9 about here

The problem just depicted vanishes if one is willing to accept the assumption of free disposal of utility. This means that if $x \in S$ and $d \leq y \leq x$, then $y \in S$. Reductions of utilities lead to points in S, whenever the point from which the reduction originates is weakly individual rational (i.e., for all $x \in S, x \geq d$). Under free disposal of utility, the Kalai–Smorodinsky solution is the unique weakly Pareto optimal point with equal relative utility gains for all players. However, this solution is not always strongly Pareto optimal. If one additionally accepts the availability of small utility transfers, then this problem disappears, too. One of the approaches that generalize the axiomatic characterization of the Kalai–Smorodinsky solution to the case of n persons is due to Imai (1983). He replaces Nash's independence axiom by two axioms, an axiom of individual monotonicity together with an axiom of independence of irrelevant alternatives other than the ideal point. The latter is a weakened version of Nash's independence condition, where situations which are being compared have identical status quo points and identical ideal points. Imai's set of axioms uniquely characterizes a lexicographic maximin solution in relative utility gains.

1.5 A philosopher's view

The ideal point plays a central role in the Kalai–Smorodinsky solution. Each coordinate in $\bar{x}(S, d)$ is such that the particular agent considered is assumed to get his or her maximal utility, while all the other players realize some individually rational feasible utility value. What importance does the ideal point have? Is it a legal or historical claim that agents can make? This is not clear at all and furthermore remember that in many cases, $\bar{x}(S, d)$ lies outside the set of feasible utility vectors.

The philosopher Gauthier views the Kalai–Smorodinsky solution as the outcome of a non–cooperative bargaining process where players have to make concessions but start, as an initial claim, from their maximally possible utility levels. The process that Gauthier considers is somewhat similar to the Zeuthen procedure of alternating concessions that led to the Nash solution, as we saw in chapter 8.3. From a conceptual point of view, Gauthier's theory is much broader than Zeuthen's proposal. Gauthier's approach has to be considered as a bargaining model of moral choice where social values are to be distributed.

Gauthier argues that a just principle for determining social values has to be based on the agreement of all individuals in a given society. The only such principle upon which rational individuals will agree, is one which is achieved through bargaining. The existence of such an agreement is necessitated by the possibility of a 'market failure', or, using Smith's metaphor, 'where the invisible hand fails to direct each person, mindful only of her own gain, to promote the benefit of all, cooperation provides a visible hand' (Gauthier (1986), p. 113). Thus, in situations in which strategic rationality leads to inefficiency, Gauthier's theory suggests that rational individuals will cooperate in order to exploit common utility gains. Gauthier (1986, p. 128) writes that 'cooperation arises from the failure of market interaction to bring about an optimal outcome because of the presence of externalities. We may then think of cooperative interaction as a visible hand which supplants the invisible hand, in order to realize the same ideal as the market provides under conditions of perfect competition'. In cooperating, each agent accepts some restrictions on his or her aim of maximizing utility. That is, each person must agree to constrain her behaviour, provided others similarly agree. Only where each person takes the interests of all others into account, can every individual achieve a utility value which is greater than his or her utility value without cooperation. Subsequently, we wish to give some more details of Gauthier's approach.

Let each bargainer i propose that the ideal payoff $\bar{x}_i$ be allocated to him or her. We know that in most cases the vector of ideal payoffs is no solution since it lies outside the feasible set. Consider any feasible outcome x which is individually rational. The concession required by person i if she agrees to x is, according to the Zeuthen formula, $(\bar{x}_i - x_i)/(\bar{x}_i - d_i)$. As explained earlier, this expression determines the ratio between the payoff person i forgoes if x is accepted in comparison to the ideal payoff and her ideal gain over the disagreement payoff. In other words, Gauthier applies Zeuthen's formula to utility differences between ideal utility values and proposed utility values and between ideal utility values and the status quo. Gauthier argues, 'Each bargainer looks upon the utility, to him, of the status quo as a minimum, and evaluates other social states in relation to that minimum' (1978, p. 246). The second modification of Zeuthen's procedure is that every concession of player i during the bargaining process is measured in relation to $\bar{x}_i - d_i$, while in Zeuthen's formula the denominator changes. This underlines the importance of the ideal point not only for the first proposals of the bargainers but also for the whole procedure.

Naturally, a person is the less willing to make a concession the larger this concession is. Therefore, consider the largest concession required for each of the possible bargaining outcomes. For any x in the set of individually rational utility vectors, the maximum concession is $\max_i (\bar{x}_i - x_i)/(\bar{x}_i - d_i)$. Since the maximum concession will obviously elicit the maximum degree of resistance to agreement, we are looking for an outcome with the least maximum degree of resistance to agreement. For Gauthier it is rational that such an outcome will be accepted. 'The person required to make the maximum concession needed to yield this outcome is more willing to concede than any person required to make the maximum concession needed to yield any other outcome' (Gauthier (1985), p. 37). Thus, the bargaining solution is the outcome with the least maximum concession. However, as Gauthier shows, the requirement that the maximum concession be minimized is equivalent to the demand that the minimum proportion of possible utility gain be maximized. The minimum proportion of possible utility gain is, of course, $\min_i (x_i - d_i)/(\bar{x}_i - d_i)$ so that according to Gauthier, given an n–person bargaining situation (S, d), $x \in S$ is the solution if and only if

$$\min_{i \in N} \frac{x_i - d_i}{\bar{x}_i - d_i} > \min_{j \in N} \frac{y_j - d_j}{\bar{x}_j - d_j}$$

for all y ($y \neq x$) that are individually rational.

Notice that the comparisons which are made with respect to proportionate gains or concessions do not presuppose any degree of interpersonal comparability of individual utilities. The Gauthier solution shares this characteristic with all the other solutions we have discussed so far. Notice also that for only two individuals, the Gauthier solution and the solution à la Kalai–Smorodinsky are identical. This is due to the fact that in the case of two agents, maximizing the minimum proportion of possible utility gains leads to an equalization of the two ratios. Unfortunately, for more than two players, Gauthier's solution concept runs into similar problems as the Kalai–Smorodinsky solution. It is not well–defined. For three agents, for example, the above (strict) inequality has to be changed into a weak inequality. Additionally, strong Pareto optimality has to be required. Only then a unique solution point is achieved (for details, see Klemisch–Ahlert (1992), pp. 87–91). For the general case of n players, this author proposed a characterization of the Gauthier solution as well (Klemisch–Ahlert (1992, chapt. 4)). In this characterization, a generalized equity axiom in relative utility gains is formulated that is reminiscent of the equity requirement used in chapter 7.3.

1.6 Kalai's egalitarian solution

In his survey on cooperative models of bargaining, Thomson (1994c) writes that 'three solutions play the central role in the theory as it appears today' (p. 1242). Therefore, let us have a brief look at the third approach, whose main distinguishing feature from Nash's solution and the Kalai–Smorodinsky solution is that it requires interpersonal comparisons of utility. Kalai (1985) discusses in greater detail an example of two bargainers 1 and 2 who are confronted with four possible allocations of money, viz. (\$0, \$0), (\$10, \$0), (\$0, \$10), and (\$0, \$1000). It is assumed that both bargainers have utility functions that increase monotonically in money. Kalai sets $u_i(\$0) = 0$ and $u_i(\$10) = 1$ for $i = 1, 2$. Now consider two bargaining situations A and B with the common status quo point $d = (0, 0)$. In A, the feasible set consists of all the lotteries among the three outcomes (\$0, \$0), (\$10, \$0), and (\$0, \$10). In B, the feasible set consists of all the lotteries between the outcomes (\$0, \$0), (\$10, \$0), and (\$0, \$1000). In utility space, situation A is mapped into $\hat{A}$ = convex hull $\{(0, 0), (1, 0), (0, 1)\}$, situation B is mapped into $\hat{B}$ = convex hull $\{(0, 0), (1, 0), (0, u_2(\$1000))\}$. Under the informational set–up CMN, agent 2's utility scale can be changed such that u_2 is transformed into $v_2 = u_2/u_2(\$1000)$. Then situation $\hat{B}$ can be described by $\hat{\hat{B}}$ = convex hull $\{(0,0),(1,0),(0,1)\}$, which in terms of utility values becomes identical to $\hat{A}$. Kalai now argues convincingly that $\hat{A}$ and $\hat{\hat{B}}$ are not identical in the sense that they should yield the same outcome. 'Player 2 stands to lose significantly more that player 1, if ... negotiations break off. Both players are aware of this fact, and it seems like a threat of player 1 to break the negotiation would have significant credibility behind it' (p. 89).

Therefore, in this section we assume that the utility scales of all individuals are comparable so that the solution should be invariant only for cases when *all* individuals' utility scales are changed linearly by the same factor.

The egalitarian solution proposed by Kalai (1977) is defined by setting, for all $(S,d) \in B^n$, $E(S,d)$ to be the maximal point of S of equal coordinates, i.e., for all $i,j \in N, E_i(S,d) - d_i = E_j(S,d) - d_j$. Figure 8.10.a illustrates the solution for the case that $d = (0,0)$ in $\mathbb{R}^2$. The egalitarian solution satisfies a monotonicity condition that is strong, since no restriction is imposed on the expansion of utility possibilities that take some S into S'. In other words, all players should benefit from any expansion of opportunities, irrespective of whether the expansion is biased in favour of one of them.

Axiom 6 (Strong Monotonicity). If (S^1,d) and (S^2,d) are any two bargaining situations in B^n such that $S^1 \subseteq S^2$), then $f(S^1,d) \leq f(S^2,d)$.

The following characterization result is closely related to a theorem developed in Kalai (1977).

THEOREM 8.3 A solution on B^n satisfies weak Pareto efficiency, symmetry, and strong monotonicity iff it is the egalitarian solution ($E(S,d)$).

The proof is easy. Therefore, we shall abstain from giving it here (see, however, Thomson (1994c) or Thomson and Lensberg (1989)).

Figure 8.10.a about here

Figure 8.10.b about here

Figure 8.10.c about here

Figure 8.10.b shows that weak Pareto efficiency in Theorem 8.3 cannot be strengthened to strict Pareto. However, there is a natural extension of solution $E(S,d)$ that is obtained by a lexicographic operation. Given $w \in \mathbb{R}^n$, let $\tilde{w} \in \mathbb{R}^n$ denote the vector obtained from w by writing its coordinates in increasing order. Given two utility vectors $x,y \in \mathbb{R}^n$, x is lexicographically larger than y if $\tilde{x}_1 > \tilde{y}_1$ or $[\tilde{x}_1 = \tilde{y}_1$, and $\tilde{x}_2 > \tilde{y}_2]$, or, more generally, for some $l \in \{1,\ldots,n-1\}$, $[\tilde{x}_1 = \tilde{y}_1, \ldots, \tilde{x}_l = \tilde{y}_l$, and $\tilde{x}_{l+1} > \tilde{y}_{l+1}]$. For $(S,d) \in B^n$, the lexicographic egalitarian solution $L(S)$ is the point of S that is lexicographically maximal (see Figure 8.10.b again for the case of two persons). The lexicographic extension $L(S)$, however, does not fulfil the axiom of strong monotonicity,

as can be seen from Fig. 8.10.c. In this situation, $S \subset S'$, but person 2 receives less utility under $L(S')$ than under $L(S)$.

Kalai (1977) shows that the egalitarian solution satisfies what he calls 'the step–by–step negotiation' condition. This means that bargaining can be done in stages without affecting the final outcome. Let (S, d) and (T, d) be two bargaining pairs with $S \subseteq T$. The bargaining agents could divide the process into two stages. In the first stage, they would agree on an outcome in (S, d), which they then use as a disagreement point for a second round of negotiations where they may agree on a new alternative in $T \backslash S$. So the egalitarian solution satisfies the property that $f(T, d) = f(S, d) + f(R)$, where R is a bargaining situation with threat point 0 and all those individually rational points that remain after agreeing upon $f(S, d)$ in the first round.

The egalitarian solution, in spite of allowing for interpersonal utility comparisons, clearly has the following welfaristic feature that the other two major solution concepts also exhibit: If two problems set up in an economic environment lead to the same set of utility possibilities, then the solution mechanism must assign solution points for the two problems which are indistinguishable in terms of utility. We shall come back to this point in the next chapter and shall discuss it from a somewhat different angle.

1.7 A short summary

The bargaining approach differs from the typical social choice approach in several respects, the existence of a status quo or disagreement point is perhaps the most significant. The analysis is entirely alone in utility space, based on von Neumann–Morgenstern utility functions. We discussed the Nash solution and the solution concept by Kalai–Smorodinsky in greater detail. Both approaches provide a unique solution point. Also, both proposals have three axioms in common. A bifurcation occurs when Nash requires a consistency condition with respect to set contraction and Kalai and Smorodinsky postulate a monotonicity condition instead. The argument behind the latter is that a solution should appropriately reflect changes in the utility possibilities of the players.

It has been shown by Harsanyi that the Nash solution of maximizing the product of utility gains over the status quo can be explained by a process of alternating concessions among the agents involved in bargaining. This idea which goes back to Zeuthen introduces elements of non–cooperative games. The Kalai–Smorodinsky solution is such that the relative utility gains of the agents are equalized. Gauthier picked up Zeuthen's idea of successive concessions and proposed a solution where the maximum concession that one of the players is required to make in order to achieve a common solution is minimal. This implies that any other bargaining outcome proposed would require a larger concession from one of the agents.

All these approaches do without any form of interpersonal comparability of utilities. Kalai's egalitarian solution presupposes that the utility scales of all persons be comparable.

1.8 Some Exercises

8.1 Let us assume that with a given amount of productive resources, two commodities in the quantities x_1 and x_2 can be produced. Let us further suppose that if all the resources go into the production of commodity 1, three units of this commodity can be produced. Likewise, if all the resources flow into the production of commodity 2, two units can be produced. All convex combinations between the points (3,0) and (0,2) are also possible by suitably dividing up the resources. Please construct the set of feasible production possibilities graphically.

Let us now assume that there are only two individuals and furthermore, that individual 1(2) is only interested in commodity 1(2) and therefore only receives commodity 1(2). The utility function of person 1 is $u_1(x_1, x_2) = 2x_1 + 2$, the utility function of person 2 is $u_2(x_1, x_2) = 3x_2 + 1$, where x_1 and x_2 stand for quantities of the first and second commodity, respectively. Construct the set of feasible utility allocations for the two individuals.

8.2 Explain why in bargaining situations such as the one in figure 8.3(b) or situation (S, d) in figure 8.4, axioms 1-3 of Nash are sufficient to determine his bargaining solution.

8.3 Given that the set S of feasible utility allocations looks like the sets S in figure 8.3 or in figure 8.10, show that the north–east corner point of the largest rectangle in S determines the coordinates of the Nash bargaining solution. Why is this so?

8.4 Take the set (S, d) of feasible utility allocations from 8.1 and calculate the utility values u_1, u_2 for the Nash bargaining solution, i.e. $f_1(S, d)$ and $f_2(S, d)$.

8.5 In the first step of the proof of theorem 8.1, (S, d) is transformed into (S', d') such that $ad + b = (0, 0)$ and $ax + b = (1, 1)$. Let $d = (4, 6)$ and $x = (7, 10)$. Determine a_1, a_2 and b_1, b_2.

8.6 Assume a bargaining situation (S, d) for two individuals where the status quo point is $d = (0, 0)$ and the set S of feasible utility vectors is given by $6u_1^2 + 3u_2^2 = 72$. Determine the Nash bargaining solution $f(S, d)$.

8.7 Construct bargaining situations where the ideal point is not outside set S and determine the Nash and the Kalai–Smorodinsky bargaining solutions for these.

8.8 The utility possibilities, i.e. set S, between persons 1 and 2 are given by $u_2 = 10 - u_1$. However, these possibilities are constrained by the fact that the highest utility level that person 2 can achieve is $\bar{u}_2 = 7$ where person 1 obtains $u_1 = 3$. For $d = (0, 0)$, depict the bargaining situation (S, d) graphically and calculate both the Nash and the Kalai–Smorodinsky solutions.

8.9 Construct two bargaining situations (T, d) and (S, d) with $S \subset T$ which show that the Kalai–Smorodinsky solution does not satisfy Nash's independence axiom.

8.10 Show that in Zeuthen's alternating offer approach of section 8.3, expressions (a) and (b) can be transformed into inequalities (a') and (b').

Recommended Reading:

Kalai, E. (1985). 'Solutions to the Bargaining Problem', in L. Hurwicz, D. Schmeidler and H. Sonnenschein (eds.), *Social Goals and Social Organization. Essays in Memory of Elisha Pazner.* Cambridge: Cambridge University Press.

Roemer, J. E. (1996). *Theories of Distributive Justice,* chapt. 2. Cambridge, Ma.: Harvard University Press.

Historical Sources:

Luce, R. D. and Raiffa, H. (1957). *Games and Decisions.* New York: John Wiley.

Nash, J. F. (1950). 'The Bargaining Problem'. *Econometrica*, 18: 155–162.

Nash, J. F. (1953). 'Two–Person Cooperative Games'. *Econometrica*, 21: 128–140.

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior.* Princeton: Princeton University Press.

More Advanced:

Thomson, W. (1994c). 'Cooperative Models of Bargaining', chapt. 35 in R. J. Aumann and S. Hart (eds.), *Handbook of Game Theory*, vol. 2. Amsterdam: North–Holland.

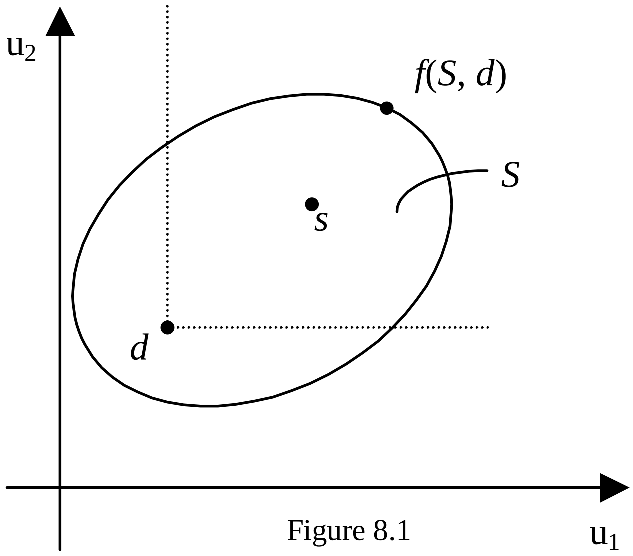

Figure 8.1

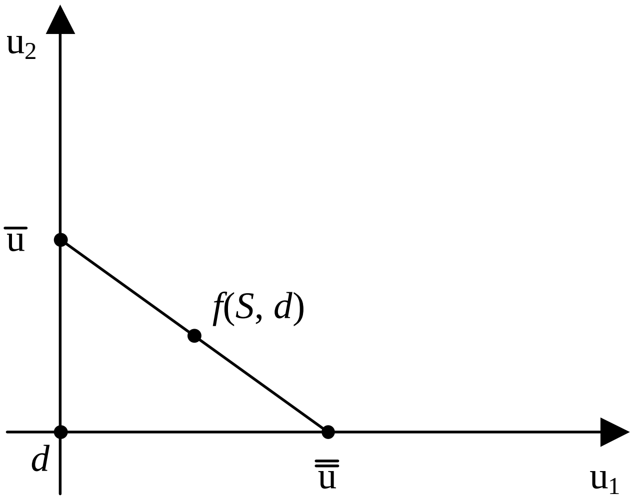

Figure 8.2.a

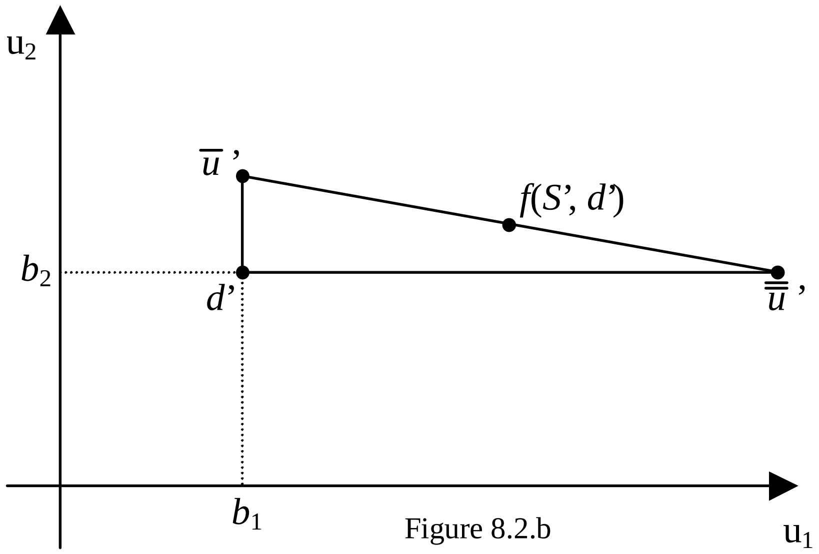

Figure 8.2.b

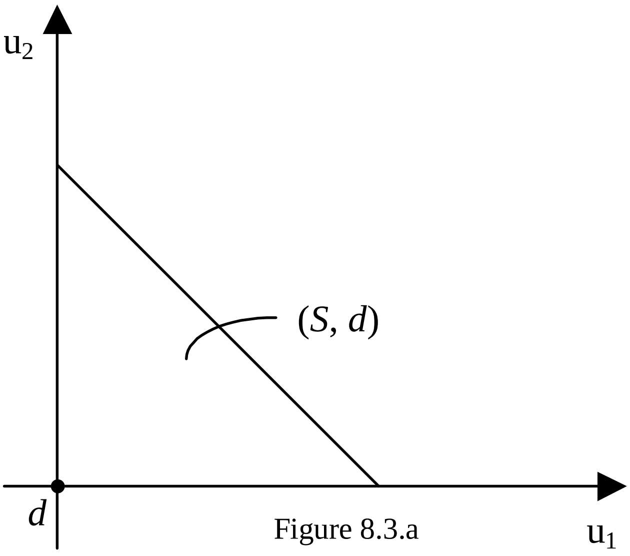

Figure 8.3.a

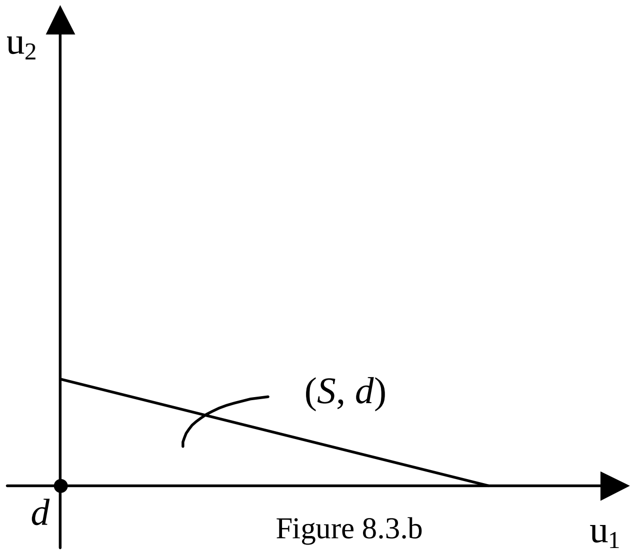

Figure 8.3.b

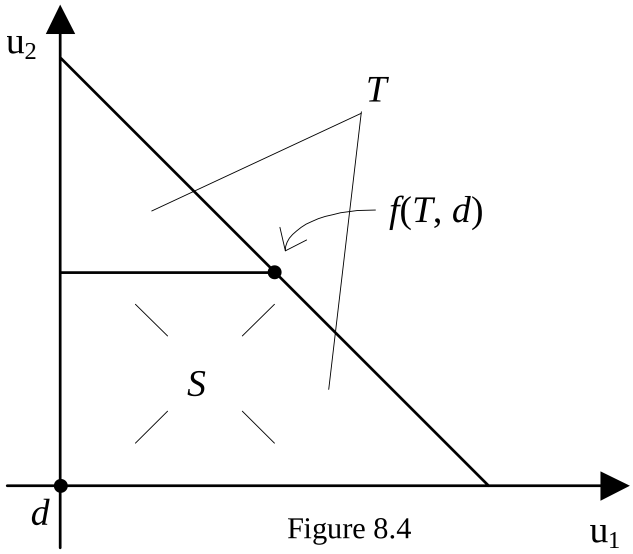

Figure 8.4

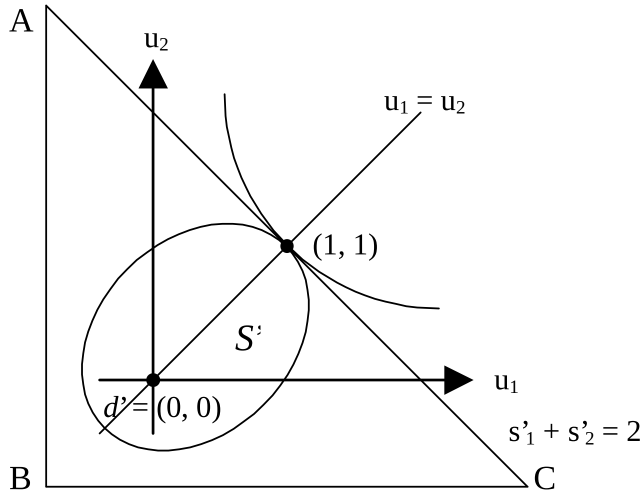

Figure 8.5

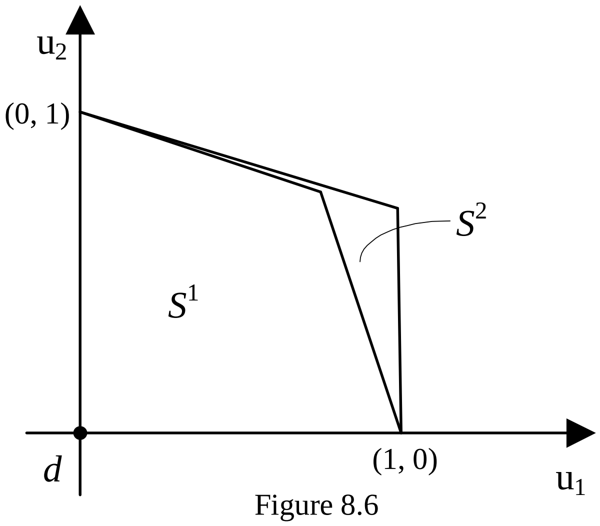

Figure 8.6

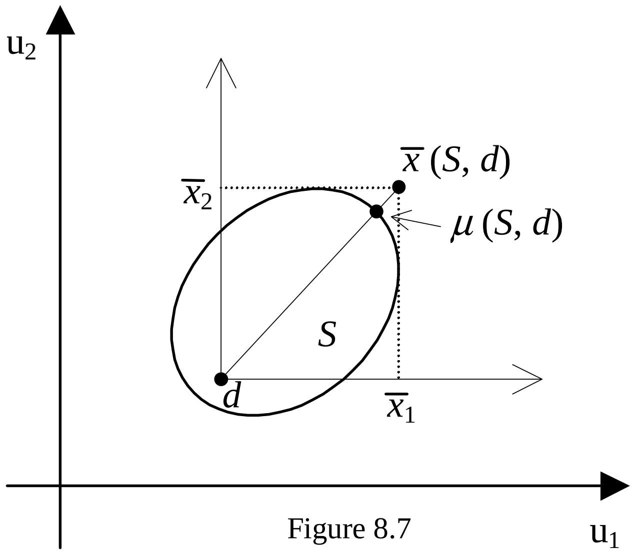

Figure 8.7

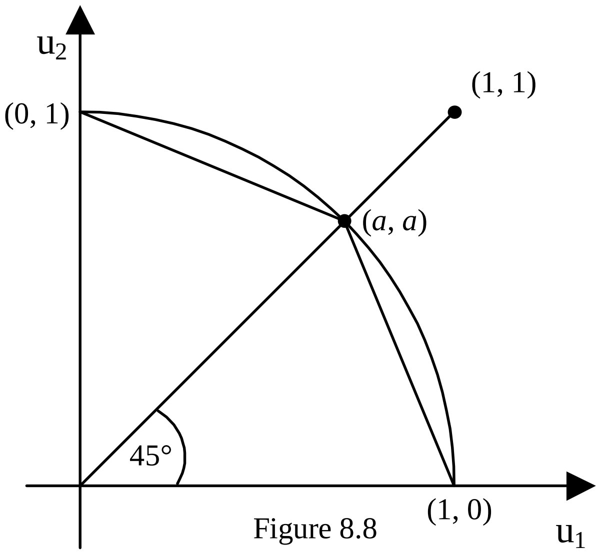

Figure 8.8

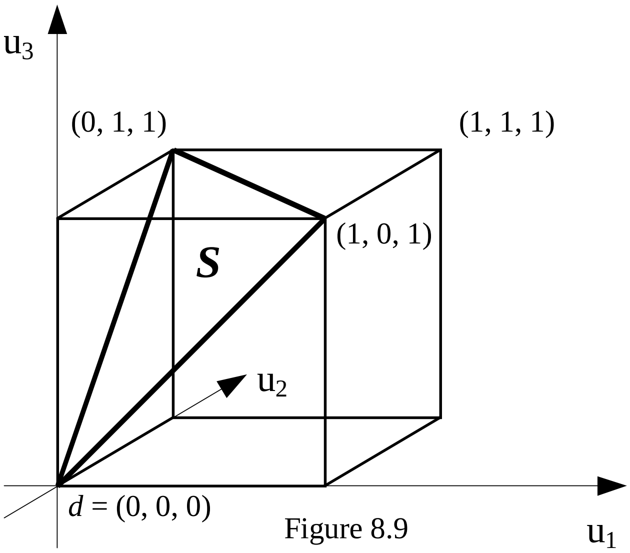

Figure 8.9

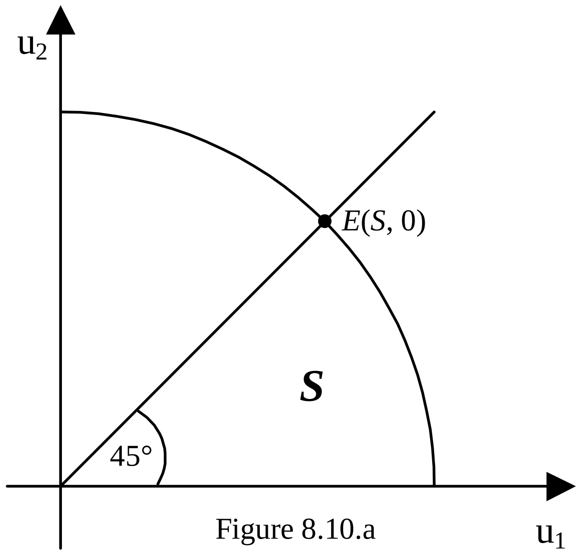

Figure 8.10.a

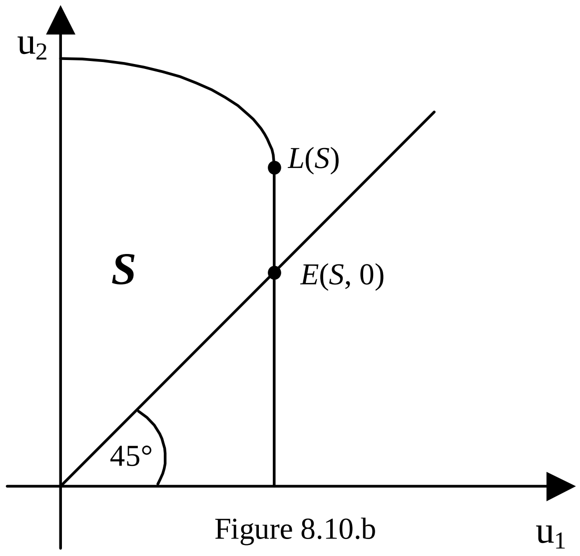

Figure 8.10.b

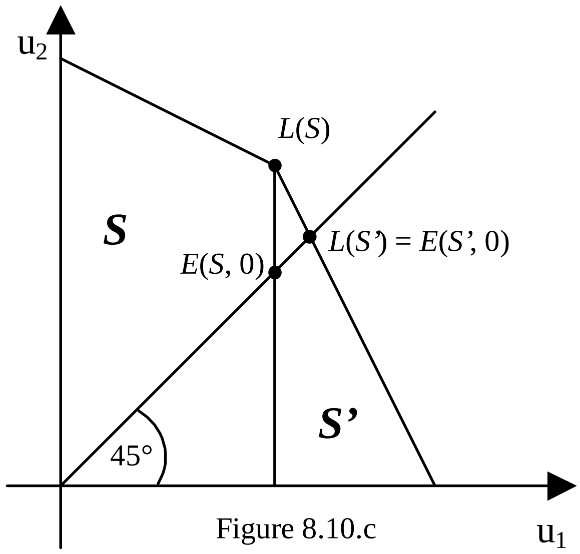

Figure 8.10.c