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Reference–Dependent Rankings of Sets in Characteristics Space

Abstract This paper uses Lancaster’s characteristics approach in order to rank sets of alternative combinations of commodity characteristics. It is assumed that there exists a reference point or reference surface from which the individual evaluates set expansions in appropriate directions. We provide an axiomatic characterization for such a case.

Keywords: ranking of sets, commodity characteristics, reference level

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by
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1 Introduction

In microeconomics, sets of alternatives are normally evaluated via families of non–intersecting indifference curves. An immediate question is: whose indifference curves? The answer is more or less unanimous: it should be the indifference curves of the concerned individual and not of somebody who pretends to know better. But there still are some queries. Should the individual’s indifference curves stem from an instantaneous or myopic utility function or preference relation or should they come from a long–range utility function which takes account of preference changes and other intertemporal aspects? An answer to this question might be that much depends on what the analysis is used for.

A quite different though related aspect concerning the evaluation of sets of alternatives is the following. It is argued that the point or points chosen via a set of indifference curves do not adequately reflect the richness of opportunities the individual experiences when making his or her choice. The relatively new freedom–of–choice literature focusses on this point and then takes off in various directions.

The standard argument to show that the richness of choice is being neglected in conventional analysis can be made very easily in the following way. Imagine that the individual considered has a simple utility function of the following kind: $u(x) = x_1 \cdot x_2$ for two commodities in quantities x_1 and x_2 (the argument can easily be generalized to higher dimensions). Let us further assume that prices are such that $p_1 = p_2 = 1$ and the budget comprises two units of money. Then the optimal, i.e. utility maximizing allocation is $x^* = (1, 1)$. Now imagine that our consumer does not have the infinitely many options of choice provided by the budget set just described but is simply offered the vector $x = (1, 1)$ for his money. Does this make a difference? The freedom–of–choice literature gave an affirmative answer and emphasized the intrinsic value of choice in great detail.

Once richness of choice is considered, an immediate question is how to measure this richness. If there is a finite number of discrete objects, counting numbers would be a possible option (Pattanaik and Xu (1990)). If richness of opportunities manifests itself in alternative budget sets in an n -dimensional Euclidean space, then a ranking rule that compares opportunity sets according to their volumes ap-

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pears as a possibility (Xu (2004)). In both approaches, there is no discrimination among objects or, put differently, objects are treated equally. In other words, these approaches focus exclusively on the quantity aspect of opportunity sets. They are non-preference-based. Of course, there are good arguments to consider the quality of alternatives as well (Sen (1991, 1993)). When poor alternatives are added to an already existing set of objects, and these poor alternatives are dominated by one or several of the existing objects in terms of quality with prices being roughly the same, then nothing valuable is added so that the richness of choice has not been increased at all. Therefore, Sen and others have argued in favour of a preference-based approach. Again, the question arises whose preferences and what kind of preference should count.

In this paper we wish to put forward an approach which is based on Lancaster's idea of looking at and evaluating characteristics (Lancaster (1966)). Our analysis will, therefore, be done in n -dimensional characteristics space. In contrast to Lancaster, we shall do without a utility function or preference relation defined on the space of characteristics. We shall require monotonicity with respect to characteristics which can be weakened to apply only to certain "directions" within the characteristics space. The individual who we consider will evaluate alternative sets of opportunities (in terms of characteristics) from a vantage point that we shall call a reference point. So in a certain sense, we are using elements from the concept of boundedly rational behaviour (Simon (1957)). Our consumer views his opportunities from the vantage point of an already realized position that could be interpreted as a status quo or – alternatively – as a point of minimal achievements below which life becomes unpleasant or miserable. The individual then explores his or her possibilities in "north-east" direction where, as just stated, all characteristics which can be attained through a purchase of various commodities are equally desirable. The expansion north-east should be made as large as possible, given the financial "capabilities" of the individual. If one collection of opportunities is finally declared to be better than another, certain comparisons among different combinations of characteristics must have been possible and the units among the different characteristics must have been rendered commensurable. This is a basic supposition we have to make. The easiest case of comparison and one that we shall consider while we go along is the one where for a given budget and given commodity prices, one set of characteristics combinations completely lies inside an alternative set so that the latter can undoubtedly be considered as better than the former. If such a situation would always or often come about, "life" would be much easier. And the instruments to make set comparisons would be much simpler. But it is our conviction that such situations will be extremely rare. Therefore, more general cases have to be tackled. Let us now go into *medias res*.

In section 2, we introduce our basic notation and some definitions. Section 3 discusses certain axiomatic properties and presents our first characterization result. Section 4 introduces the concept of a reference level that contains more

than one point. Section 5 discusses the case where monotonicity only holds inside certain directed cones. We end with some concluding remarks in section 6.

2 Basic Notation and Definitions

Let \mathbb{R}_+ be the set of all non-negative real numbers, \mathbb{R}_{++} be the set of all positive numbers, \mathbb{R}_+^n be the n -fold Cartesian product of \mathbb{R}_+ , and \mathbb{R}_{++}^n be the n -fold Cartesian product of \mathbb{R}_{++} . The vectors in \mathbb{R}_+^n will be denoted by x, y, z, a, b, \dots , and are interpreted as vectors of characteristics (Lancaster (1966)). For all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}_+^n$, define $x \geq y$ as $x_i \geq y_i$ for all $i = 1, \dots, n$, $x > y$ when $x \geq y$ and $x \neq y$, and $x \gg y$ when $x_i > y_i$ for all $i = 1, \dots, n$.

There are perhaps several ways to measure the achievements that an agent makes when moving from a vector of characteristics x to another vector y . In this paper, we shall use the notion of a distance function to capture the progress made by the individual. For this purpose, we shall focus¹ on the commonly used Euclidean distance $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

At any given point of time, the set of all vectors that may be available to the individual is a subset of \mathbb{R}_+^n . Such a set will be called the individual's *characteristics set*. We will use A, B, C , etc. to denote such sets.

Our concern in this paper is to rank different characteristics sets in terms of the achievements that they offer to the individual. In particular, we confine our attention to sets that are

(2.1) *compact*: a set $A \subseteq \mathbb{R}_+^n$ is compact iff A is closed and bounded,

(2.2) *convex*: a set $A \subseteq \mathbb{R}_+^n$ is convex iff, for all $x, y \in \mathbb{R}_+^n$ and all $\alpha \in [0, 1]$, if $x, y \in A$, then $\alpha x + (1 - \alpha)y \in A$,

(2.3) *star-shaped*: a set $A \subseteq \mathbb{R}_+^n$ is star-shaped iff, for all $x \in \mathbb{R}_+^n$ and all $t \in [0, 1]$, if $x \in A$, then $tx \in A$.

Let \mathcal{K} be the set of all characteristics sets that are compact, convex and star-shaped. For all $A, B \in \mathcal{K}$, we write $A \subseteq B$ for “ A being a subset of B ” and $A \subset B$ for “ A being a proper subset of B ”.

For all $A, B \in \mathcal{K}$ and all $x^* \in \mathbb{R}_+^n$, let $A >_{x^*} B$ denote: [whenever $x^* \in B$ and given d , there is a neighborhood, $X(x^*, \epsilon_d, d) = \{x \in \mathbb{R}_+^n : x \geq x^*, d(x, x^*) \leq \epsilon_d\}$ where $\epsilon_d > 0$ of x^* such that $X(x^*, \epsilon_d, d) \subseteq A$] and [for all $b \in B$ with $b > x^*$, there exists $a \in A$ such that $a \gg b$]. Let $x^0 \in \mathbb{R}_{++}^n$ be a vector of achievements below which the individual's situation is judged to be unsatisfactory. Points below x^0 may be considered unsatisfactory because they offer combinations of

¹In a related approach which evolves in the space of functionings, Gaertner and Xu (2008) consider a whole class of distance functions.

characteristics which are less than “minimal” or such combinations would just simply be below the status quo which is viewed as a base line. We shall call x^0 a reference point. Throughout this section and Section 3, we assume that x^0 is fixed. For all $t \geq 0$, define

$$X(x^0, t) = \{x \in \mathbb{R}_+^n : x \geq x^0, d(x, x^0) \leq t\}.$$

Scalar t measures the distance between two vectors in characteristics space, according to the Euclidean distance function d . This, of course, presupposes that we can quantify each of the characteristics appropriately so that there is a measurement scale common to all characteristics considered.

For all $A \in \mathcal{K}$, let

$$r(A) = \begin{cases} -1 & \text{if } x^0 \notin A, \\ \max_t \{t \in \mathbb{R}_+ : X(x^0, t) \subseteq A\} & \text{if } x^0 \in A. \end{cases}$$

We note that, for all $A \in \mathcal{K}$, if $x^0 \notin A$, then $r(A) = -1$ and if $x^0 \in A$, then $r(A) \geq 0$.

Figure 1 depicts the maximal $t \in \mathbb{R}_+$ for two sets of characteristics A and B when $x^0 \in A \cap B$ and given the Euclidean distance function d .

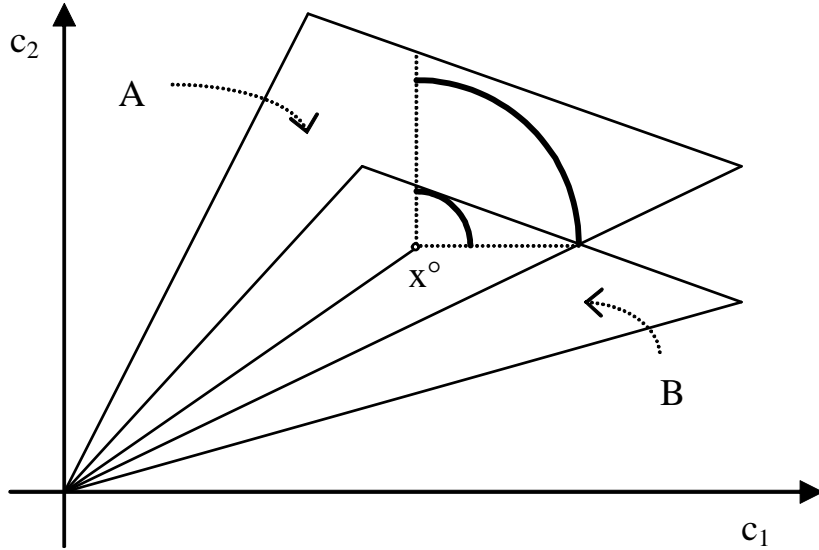


Figure 1: comparison of two characteristics sets A and B

Let \succeq be a binary relation over \mathcal{K} that satisfies *reflexivity*: [for all $A \in \mathcal{K}$, $A \succeq A$], *transitivity*: [for all $A, B, C \in \mathcal{K}$, if $A \succeq B$ and $B \succeq C$ then $A \succeq C$], and

completeness: [for all $A, B \in \mathcal{K}$ with $A \neq B$, $A \succeq B$ or $B \succeq A$]. Such a binary relation is called an *ordering*. The intended interpretation of \succeq is the following: for all $A, B \in \mathcal{K}$, $[A \succeq B]$ will be interpreted as “*the degree of achievements in terms of characteristics offered by A is at least as great as the degree of achievements offered by B*”. \succ and \sim , respectively, are the asymmetric and symmetric part of \succeq , and are defined as follows: for all $A, B \in \mathcal{K}$, $A \succ B$ iff $A \succeq B$ and not $B \succeq A$, and $A \sim B$ iff $A \succeq B$ and $B \succeq A$.

3 Axiomatic Properties and First Result

This section discusses several axioms that are used for characterizing a measure of well-being in terms of achievements of characteristics. They are similar to those proposed in Gaertner and Xu (2008a) for the exercise of ranking capability sets in terms of living standards.

Definition 3.1. \succeq over \mathcal{K} satisfies

- (3.1.1) **Monotonicity** iff, for all $A, B \in \mathcal{K}$, if $B \subseteq A$ then $A \succeq B$.
- (3.1.2) **Betweenness** iff, for all $A, B \in \mathcal{K}$, if $A \succ B$ with $x^0 \in A \cap B$, then there exists $C \in \mathcal{K}$ such that $C \succ_{x^0} B$ and $A \succ C \succ B$.
- (3.1.3) **Dominance** iff, for all $A, B \in \mathcal{K}$, if $x^0 \notin B$, then $A \succeq B$, and furthermore, if $x^0 \in A$, then $A \succ B$.
- (3.1.4) **Symmetric Expansion from the Reference Point** iff, for all $A, B \in \mathcal{K}$, if there exists $t > 0$ such that $X(x^0, t) \cap A = X(x^0, t)$, and $B \cap X(x^0, t) \subset X(x^0, t)$, then $A \succ B$.

The intuition behind Monotonicity is simple and easy to explain. It requires that whenever B is a subset of A , then A is ranked at least as high as B in terms of achievements. Various versions of Monotonicity have been proposed in the literature; see among others, Pattanaik and Xu (2007), and Xu (2002; 2003). Betweenness requires that when A is judged to offer more achievements than B relative to the reference vector x^0 , there must exist a set C such that $C \succ_{x^0} B$ and A offers more achievements than C , which in turn offers more achievements than B . Dominance requires that whenever the reference vector x^0 is not attainable in B , the level of achievements offered by B cannot be higher than that offered by any other set A , and furthermore, if the reference vector x^0 is attainable under A , then A offers a higher level of achievements than B . Symmetric Expansion from the Reference Point requires that, for two sets A and B , whenever A results from progress made, according to distance function d , in all dimensions of characteristics vectors, while B does not offer this particular kind of progress, the level of achievements under A is judged to be

higher than that offered by B . The four axioms above allow us to rank sets of characteristics. Notice that this axiom system is compatible with the traditional axiom of monotonicity which requires that for two vectors c and c' , let's say, such that $c > c'$, the former vector of characteristics is preferable to the latter.

We now present the following axiomatic characterization of a ranking of achievements defined as follows: for all $A, B \in \mathcal{K}$,

$$A \succeq^r B \Leftrightarrow r(A) \geq r(B).$$

Theorem 3.1. Suppose that \succeq over \mathcal{K} is an ordering. Then, \succeq satisfies Monotonicity, Betweenness, Dominance, and Symmetric Expansion from the Reference Point if and only if $\succeq = \succeq^r$.

This result is essentially the same result that we obtained in Gaertner and Xu (2008, Theorem 4.1) for a different context; its proof will therefore be omitted.

4 A Generalization: A Reference Set

Up to this point, it has been assumed that the reference level of the individual is just a single point. What happens when this reference level extends to more than one point? In the sequel, we shall assume that the reference level becomes a surface $S(c) = \{x \in \mathbb{R}_+^n : f(x) = c\}$, where $c > 0$ and $f(x)$ being a “smooth” and strictly increasing function from \mathbb{R}_+^n to \mathbb{R}_+ , with $S(c) \neq \emptyset$, $L(S(c)) = \{x \in \mathbb{R}_+^n : f(x) \leq c\}$ being convex and $z \in S(c)$ being such that $z \gg 0$. The convexity of $L(S(c))$ allows for trade-offs among characteristics; it can be seen as a technical rate of substitution between characteristics. For example, as far as vacation resorts are concerned, a higher degree of tranquility combined with lower food quality may be considered equivalent (in terms of a reference level) to a lower degree of tranquility combined with higher food quality. In terms of nutrition, different combinations of vitamins, calcium and protein may serve as equivalent levels of reference. The “area” of this surface may vary from individual to individual.

Given the reference set $S(c)$, let $x^i \in S(c)$. For any $t \geq 0$, define $X(S(c), x^i, t) := \{x \in \mathbb{R}_+^n : x \geq x^i, d(x, x^i) \leq t\}$. For any set $A \in \mathcal{K}$ and any $x^i \in S(c)$, let $t(A, x^i) = \max\{t : X(S(c), x^i, t) \subseteq A\}$. So $t(A, x^i)$ is the maximal extension “north-east” starting from $x^i \in S(c)$ such that the “quarter-ball” is still contained in A . For any set A , let $t^*(A) = \min_{x^i \in S(c)} t(A, x^i)$ denote the minimum of all the maximal extensions along $S(c)$ for set A .

For all $A \in \mathcal{K}$, let

$$r(A, S(c)) = \begin{cases} t^*(A) & \text{if } S(c) \subseteq A \\ -1 & \text{if } x^* \notin A \text{ for some } x^* \in S(c). \end{cases}$$

We note that, for all $A \in \mathcal{K}$, if $x^* \notin A$ for some $x^* \in S(c)$, then $r(A, S(c)) = -1$ and if $S(c) \subseteq A$, then $r(A, S(c)) \geq 0$.

With the help of the above definition, we consider the following measure of achievements of characteristics to be characterized in this section. Let \succ^* over \mathcal{K} be defined as follows:

$$\text{For all } A, B \in \mathcal{K}, A \succeq^* B \Leftrightarrow r(A, S(c)) \geq r(B, S(c)).$$

Figures 2 and 3 try to depict two interesting cases. In Figure 2, one can see that $t^*(A) < t^*(B)$ showing that $B \succ^* A$. Figure 3 illustrates a situation in which $A \succ^* B$ since $t^*(A) > t^*(B)$.

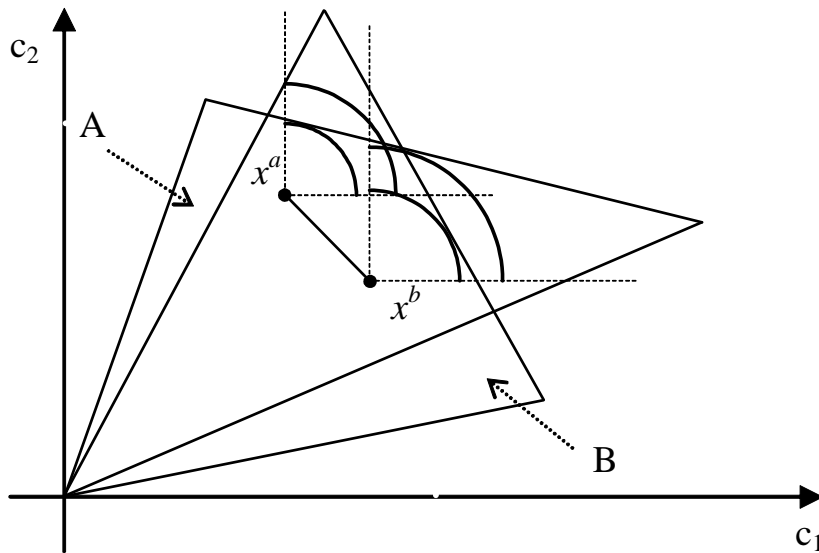


Figure 2

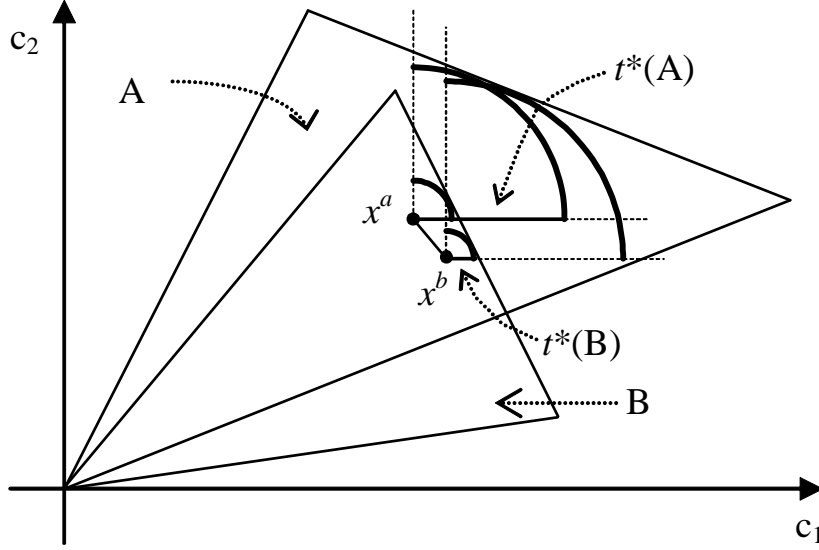


Figure 3

Before presenting axiomatic properties for characterizing the above measure, we introduce a notion first. For all $A, B \in \mathcal{K}$, let $A \succ_{S(c)} B$ denote: $S(c) \subseteq A \cap B$, $A \succ_x B$ for all $x \in S(c)$.

Definition 4.1. \succeq over \mathcal{K} satisfies

- (4.1.1) **S-Betweenness** iff, for all $A, B \in \mathcal{K}$, if $A \succ B$ with $S(c) \in A \cap B$, then there exists $C \in \mathcal{K}$ such that $C \succ_{S(c)} B$ and $A \succ C \succ B$.
- (4.1.2) **S-Dominance** iff, for all $A, B \in \mathcal{K}$, if $x^* \notin B$ for some $x^* \in S(c)$, then $A \succeq B$, and furthermore, if $S(c) \subseteq A$, then $A \succ B$.
- (4.1.3) **Symmetric Expansion from the Reference Set** iff, for all $A, B \in \mathcal{K}$, if there exists $t > 0$ such that [for all $x^i \in S(c)$, $X(S(c), x^i, t) \cap A = X(S(c), x^i, t)$, and $B \cap X(S(c), x^i, t) \subset X(S(c), x^i, t)$ for some $x^i \in S(c)$], then $A \succ B$.

Theorem 4.1 Suppose that \succeq over \mathcal{K} is an ordering and given the reference set $S(c)$. Then \succeq satisfies Monotonicity, S-Betweenness, S-Dominance and Symmetric Expansion from the Reference Set if and only if $\succeq = \succeq^*$.

Proof. It can be checked that \succeq^* is an ordering and satisfies Monotonicity, S-Betweenness, S-Dominance and Symmetric Expansion from the Reference Set. We now show that if \succeq over \mathcal{K} satisfies the above four axioms, then $\succeq = \succeq^*$.

Let $A, B, \in \mathcal{K}$. We consider the following cases:

- (i) We first note that, if $A = B$, then $A \sim B$ follows from reflexivity of \succeq directly.
- (ii) Consider that for some $x^*, y^* \in S(c)$, $x^* \notin A$ and $y^* \notin B$. Since $x^* \notin A$, by S-Dominance, we have $B \succeq A$. Similarly, by S-Dominance, from $y^* \notin B$, we have $A \succeq B$. Therefore, in this case $A \sim B$.
- (iii) Suppose that $S(c) \subseteq A$ but for some $y^* \in S(c)$, $y^* \notin B$. Then, by S-Dominance, it follows that $A \succ B$.
- (iv) Now, suppose that $[S(c) \subseteq A \cap B$ and $r(A, S(c)) > r(B, S(c)) \geq 0]$. We want to show that $A \succ B$. Then there exists $t > 0$ such that $t = t^*(A)$, $X(S(c), t^*(A)) \cap A = X(S(c), t^*)$ and $B \cap X(S(c), t^*(A)) \subset X(S(c), t^*(A))$. By Symmetric Expansion from the Reference Set, we obtain $A \succ B$.
- (v) Finally, suppose that $[r(A, S(c)) \geq r(B, S(c)) \geq 0]$, then $(A \succeq B)$. Clearly, $S(c) \subseteq A \cap B$. To show that $(A \succeq B)$ holds in this case, we use a proof by contradiction. Suppose not, by the completeness of \succeq , suppose that $B \succ A$ holds true. Then, by S-Betweenness, there exists $C \in \mathcal{K}$ such that $C \succ_{S(c)} A$ and $B \succ C \succ A$. Since $C \succ_{S(c)} A$, there exists $C' \in \mathcal{K}$ such that $C' \subseteq C$ and there exists $t > 0$ such that $t = t^*(C')$, $X(S(c), t^*(C')) \cap C' = X(S(c), t^*(C'))$ and $X(S(c), t^*(C')) \cap A \subset X(S(c), t^*(C'))$. Then, $r(C', S(c)) > r(A, S(c))$. Note that $r(A, S(c)) \geq r(B, S(c))$. Therefore, $r(C', S(c)) > r(B, S(c))$. From (iv) above, $C' \succ B$. On the other hand, from $C' \subseteq C$, by Monotonicity, $C \succeq C'$. The transitivity of \succeq implies that $C \succ B$, a contradiction with $B \succ C$, which was established a few lines above. Therefore, $(A \succeq B)$.

Since \succeq is complete, (i) – (v) establish the proof of Theorem 4.1. ■

5 Monotonicity Inside Cones

In the preceding sections, it was assumed that every expansion northeast of x^0 is exactly as desirable as every other expansion northeast of the same length or size. This makes perfect sense in the case of elementary characteristics which are vital for subsistence. But we believe that there are many situations where characteristics in certain proportions are viewed as more appropriate than other combinations. Consider a well-balanced nutrition for children or older people that is designed to establish certain proportions of various vitamins together with certain quantities of protein and calcium, let's say. Or imagine an individual who ponders over the "right" mode of transportation where factors such as speed, punctuality, comfort and accessibility should be adequately combined.

Or consider alternative vacation projects that for one person should combine properties such as being adventurous, exclusive and offering exquisite food, while another person would aspire to achieve a certain proportion of nature, culture and access to healthy food. All these different aspects are indeed characteristics in Lancaster's sense.

More technically, let $X(x^0, \delta)$ be a right cone with angle δ and vertex x^0 . Note that, if $\delta = 0$, then there is a unique direction. The width of angle δ can be due to an individual's uncertainty about the "right" proportion of characteristics or the outgrowth of a probability distribution over "suitable" proportions of characteristics.

Let $t > 0$ and for our given distance function d , we define $X_\delta(x^0, t, d)$ as follows:

$$X_\delta(x^0, t, d) = \{x_\delta \in X(x^0, \delta) : x_\delta \geq x^0, d(x_\delta, x^0) \leq t\}$$

For purposes of illustration and given the Euclidean distance function d , Figure 4 shows two possible angles of expansion in characteristics space. As will be clear from the following analysis, maximal expansion will again be our criterion. In Figure 4, the direction of the desired angle will not matter in a set comparison between A and B . Given the position of the reference point x^0 , set A will always be better than set B . Figure 5 depicts a situation, where the angle has shrunk to a unique direction. If vectors x_A and x_B in this figure either lie on a line perpendicular to x^0 or on a line horizontal to x^0 , the person considered would only be interested in one of the characteristics, given x^0 . One could interpret this as partial satiation, i.e., satiation with respect to one of the two characteristics.

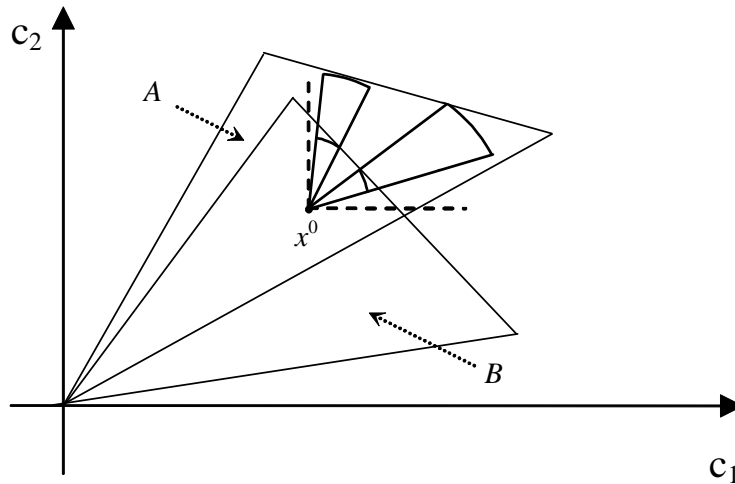


Figure 4

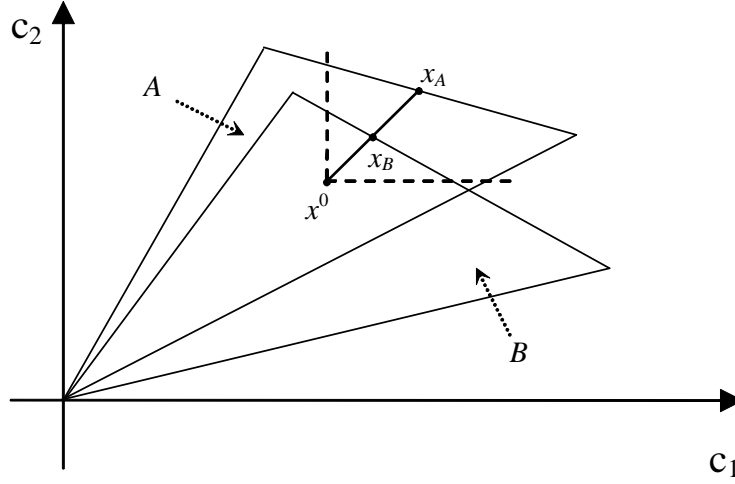


Figure 5

Let us define, for all $A \in \mathcal{K}$,

$$r_\delta(A, d) = \begin{cases} -1 & \text{if } x^0 \notin A, \\ \max_t \{t \in \mathbb{R}_+ : X_\delta(x^0, t, d) \subseteq A\} & \text{if } x^0 \in A. \end{cases}$$

Here, $r_\delta(A, d)$ measures the maximal extension of sectors along the given cone $X(x^0, \delta)$.

We now introduce axioms 3.1.1 to 3.1.3 from section 3 and replace axiom 3.1.4 by an axiom that considers asymmetric expansions from the reference point. We shall say that \succeq over \mathcal{K} satisfies

- (5.1) **Asymmetric Expansion from the Reference Point** iff, for all $A, B \in \mathcal{K}$, if there exists $t_d > 0$ such that $X_\delta(x^0, t_d, d) \cap A = X_\delta(x^0, t_d, d)$, and $B \cap X_\delta(x^0, t_d, d) \subset X(x^0, t_d, d)$, then $A \succ B$.

Asymmetric expansion is in the spirit of axiom 3.1.4. It requires that, for two sets A and B , whenever A results from progress made inside the cone of desirable improvement, measured by distance function d , while B does not offer

this particular kind of progress, the level of achievements under A is judged to be higher than that offered by B .

The four axioms now allow us to rank sets of characteristics as follows: for all $A, B \in \mathcal{K}$,

$$A \succeq_{\delta}^r B \Leftrightarrow r_{\delta}(A, d) \geq r_{\delta}(B, d).$$

Theorem 5.1. Suppose that \succeq over \mathcal{K} is an ordering and given a distance function d and given a right cone $X(x^0, \delta)$. Then, \succeq satisfies Monotonicity, Betweenness, Dominance, and Asymmetric Expansion from the Reference Point if and only if $\succeq = \succeq_{\delta}^r$.

Proof. It can be checked that the achievement ranking \succeq introduced in section 2 is an ordering and satisfies the four axioms. We now have to show that if \succeq over \mathcal{K} satisfies these axioms, then \succeq is the ranking proposed above.

- (i) We first show that, for all $A, B \in \mathcal{K}$, if $A = B$ or $[r_{\delta}(A, d) = r_{\delta}(B, d) = -1]$, then $A \sim B$. Let $A, B \in \mathcal{K}$. When $A = B$, by reflexivity of \succeq , $A \sim B$ follows easily. Consider next that $[r_{\delta}(A, d) = r_{\delta}(B, d) = -1]$, that is, $x^0 \notin A$ and $x^0 \notin B$. Since $x^0 \notin A$, by Dominance, it follows that $B \succeq A$. Similarly, by Dominance and from $x^0 \notin B$, it follows that $A \succeq B$. Therefore $A \sim B$.
- (ii) Second, we show that for all $A, B \in \mathcal{K}$, if $r_{\delta}(A, d) \geq 0 > r_{\delta}(B, d) = -1$, then $A \succ B$. Note that, in this case, it must be the case that $x^0 \in A$ and $x^0 \notin B$. By Dominance, $A \succ B$ follows easily.
- (iii) Third, we show that for all $A, B \in \mathcal{K}$, if $[r_{\delta}(A, d) > r_{\delta}(B, d) \geq 0]$, then $A \succ B$. Let $A, B \in \mathcal{K}$ be such that $[r_{\delta}(A, d) > r_{\delta}(B, d) \geq 0]$. Then there exists $t_d > 0$ such that $X_{\delta}(x^0, t_d, d) \cap A = X_{\delta}(x^0, t_d, d)$ and $B \cap X_{\delta}(x^0, t_d, d) \subset X_{\delta}(x^0, t_d, d)$. By Asymmetric Expansion from the Reference Point, we obtain $A \succ B$.
- (iv) We next show that, for all $A, B \in \mathcal{K}$, if $[r_{\delta}(A, d) \geq r_{\delta}(B, d)]$, then $(A \succeq B)$. Let $A, B \in \mathcal{K}$, and $[r_{\delta}(A, d) \geq r_{\delta}(B, d)]$. Note that if $r_{\delta}(B, d) = -1$, then $x^0 \notin B$. We have already dealt with this situation in (i) and (ii). Therefore, we assume that $r_{\delta}(B, d) \geq 0$. Hence, $x^0 \in A \cap B$. To show that $(A \succeq B)$ holds in this case, we use a proof by contradiction. Suppose not, by the completeness of \succeq , suppose that $B \succ A$ holds true. Then, by Betweenness, there exists $C \in \mathcal{K}$ such that $C \succ_{x^0} A$ and $B \succ C \succ A$. Since $C \succ_{x^0} A$, there exists $C' \in \mathcal{K}$ such that $C' \subseteq C$ and there exists $t_d > 0$ such that $X_{\delta}(x^0, t_d, d) \cap C' = X_{\delta}(x^0, t_d, d)$ and $X_{\delta}(x^0, t_d, d) \cap A \subset X_{\delta}(x^0, t_d, d)$. Then, $r_{\delta}(C', d) > r_{\delta}(A, d)$. Note that $r_{\delta}(A, d) \geq r_{\delta}(B, d)$. Therefore, $r_{\delta}(C', d) > r_{\delta}(B, d)$. From (iii) above, $C' \succ B$. On the other hand, from $C' \subseteq C$, by Monotonicity, $C \succeq C'$. The transitivity of \succeq implies that $C \succ B$, a contradiction with $B \succ C$, which was established a few lines above. Therefore, $(A \succeq B)$.

Note that \succeq is complete. Then, (i) – (iv) establish the proof of Theorem 5.1.

■

6 Concluding Remarks

In this paper we used Lancaster’s characteristics approach in order to rank sets of alternative characteristics combinations. We deviated from the standard approach which considers convex sets of characteristics combinations and then imposes a utility function that is to be maximized in the space of characteristics.

We introduced a reference vector as our point of orientation in order to measure the richness of choice and then established a ranking over different sets of characteristics (see also Gaertner and Xu (2008) for a class of measures in the space of functionings). We also generalized the approach to a setting in which the reference level may be more than one point. We assumed that it would be a smooth surface with its “lower contour set” being convex. In such a situation, an extension “north–east” can be measured in different ways. In the present version, we focused on the minimal–maximal extension north–east. Other types are, of course, possible.

Note that in the present version if the reference surface $S(c)$ is not fully contained neither in set A nor in set B , then according to S –Dominance, the two sets are equivalent. This can, of course, be modified in order to be more discriminatory. But in order to do this, we would have to introduce some additional structure, which we leave for future research. Only one case can be easily dealt with in the current set–up. If $A \cap S(c) \neq \emptyset$ and $B \cap S(c) = \emptyset$, we can require that $A \succ B$.

We have then considered a restricted type of monotonicity that considers improvements within directed cones. The idea behind this proposal is the observation that very often consumer purchases follow the notion of balancedness or the “right” proportion.

Once the agent has determined, among various alternative sets of characteristics, the best, she will, within the angle of desirable improvements, go to the northeast frontier of the feasible set of characteristics and pick a point there. The width of the angle determines an interval of feasible substitutability along the frontier. If the agent is sure about the right proportion, she will pick a unique point. If she is undecided within the angle, she will pick some point along the interval. If the angle shrinks to a single line, a unique point along the northeast boundary of the feasible set of characteristics combinations is chosen. There is no room anymore for substitution among the characteristics.

The idea of directed cones enables us to deal with the phenomenon of satiation. It may very well be the case that with higher income and a higher standard of living, some of the characteristics become less important for the consumer so that at some point, a satiation level is reached for these characteristics. In this case,

the cone of desirable expansion will “lose” some dimensions. With respect to the other dimensions where there is no satiation, one would again proceed as much as possible in the still desirable directions. This aspect deserves further elaboration.

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